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Author
Flores, Cynthia Vanessa

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On decay properties of solutions to the IVP for the Benjamin-Ono equation

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by

Cynthia Vanessa Flores

Committee in Charge:

Professor Gustavo Ponce, Chair

Professor Thomas Sideris

Professor Hector Ceniceros

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The Dissertation of
Cynthia Vanessa Flores is approved:

Professor Thomas Sideris

Professor Hector Ceniceros

Professor Gustavo Ponce, Committee Chairperson

May 2014

On decay properties of solutions to the IVP for the Benjamin-Ono equation

For Jadiel

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Curriculum Vitæ

Cynthia Vanessa Flores

Education

Ph.D. Mathematics, University of California Santa Barbara, expected 2014

MS Mathematics, California State University- Northridge, 2009
Thesis: Manifolds with nonnegative isotropic ricci curvature

BA, Mathematics, California State University-Northridge, 2007
Report: Dressing actions and solitons

Publication

C.Flores. On decay properties of solutions to the IVP for the Benjamin-Ono equation. *J. Dynam. Differential Equations,* 25(4):907923, 2013.

Awards and special accomplishments

SIAM Recognition Award, UCSB, 20014

UCSB Academic Senate Graduate Student Travel Award, 2014

AMS Graduate Student Travel Award, 2013

Department of Mathematics Graduate Merit Award, UCSB, 2012

Department of Mathematics Merit Fellowship, CSUN, 2009

Association of Retired Faculty Memorial Award for Research, CSUN, 2009

NSF GK-12 Fellowship, Department of Mathematics, CSUN, 2008

2008 Sally Casanova Pre-Doctoral Scholarship, CSUN, 2008

NSF M.C.T.P Fellowship, Department of Mathematics, CSUN, 2007

Abel Scholarship, Department of Mathematics, CSUN, 2006

Abstract

On decay properties of solutions to the IVP for the Benjamin-Ono equation

Cynthia Vanessa Flores

In recent years there has been an intense activity in the study of harmonic analysis and its application to partial differential equations (PDEs). The tools of harmonic analysis assist in the discovery of important properties of certain PDEs; amid these PDEs one finds the study of nonlinear dispersive equations.

In particular, the problems of establishing local and global well-posedness under minimal regularity requirement of the given data, the long-time behavior of local solutions to these models, scattering, blow-up, and the unique continuation properties of the solutions have been extensively studied.

Among the systems considered one finds the Korteweg-de Vries equation, the Schrödinger system and the Benjamin-Ono equation, all occurring in different physical problems, mainly nonlinear wave propagation. Furthermore, under certain circumstances, these all admit solitary wave solutions called *traveling waves*, which have important applications in fiber optics, magnetics and genetics.

In this thesis we investigate unique continuation properties of solutions to the initial value problem associated to the Benjamin-Ono equation given by

$$\begin{cases} \partial_t u + \mathcal{H}\partial_x^2 u + u\partial_x u = 0, \quad x, t \in \mathbb{R} \\ \\ u(x,0) = u_0(x) \end{cases}$$

with $\mathcal{H}$ denoting the Hilbert transform

$$\mathcal{H}f(x) = \frac{1}{\pi}\,\text{p.v.}(\frac{1}{x} * f)(x) = \frac{1}{\pi}\lim_{\epsilon \downarrow 0} \int_{\epsilon<|y|<\frac{1}{\epsilon}} \frac{f(x-y)}{y} dy$$

$$= -i\,(\text{sgn}(\xi)\,\widehat{f}(\xi))^{\vee}(x).$$

in weighted Sobolev spaces $Z_{s,r} = H^s(\mathbb{R}) \cap L^2(|x|^{2r}dx)$ for $r, s \in \mathbb{R}$, and $s \geq 1$, $s \geq r$. More precisely, we prove that the uniqueness property based on a decay requirement at three times can not be lowered to two times even by imposing stronger decay on the initial data.

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Chapter 1

Introduction

This dissertation is concerned with the initial value problem (IVP) associated to the Benjamin-Ono (BO) equation

$$\begin{cases} \partial_t u + \mathcal{H}\partial_x^2 u + u\partial_x u = 0, \quad x, t \in \mathbb{R} \\ u(x,0) = u_0(x) \end{cases} \tag{1.0.1}$$

with $\mathcal{H}$ denoting the Hilbert transform

$$\begin{aligned} \mathcal{H}f(x) &= \frac{1}{\pi}\,\text{p.v.}(\frac{1}{x} * f)(x) = \frac{1}{\pi}\lim_{\epsilon\downarrow 0}\int_{\epsilon<|y|<\frac{1}{\epsilon}} \frac{f(x-y)}{y}dy \\ &= -i\,(\operatorname{sgn}(\xi)\,\widehat{f}(\xi))\check{}\,(x). \end{aligned} \tag{1.0.2}$$

The Benjamin-Ono equation describes long internal waves in a stratified medium of infinite depth. In the context of propagation of nonlinear waves, it was first deduced by B. Benjamin [3] and then by H. Ono [22]. Later, the system was shown to be completely integrable (see [9], [6]). Thus, it follows a historical pattern displayed by systems like the well known Korteweg and de-Vries (KdV) equation (modeling surface waves), modified KdV (mKdV), and cubic nonlinear Schrödinger (NLS) equation in 1-dimension. Again, the BO was first studied as a model in propagation of waves and later shown to be completely integrable.

We note that the dispersive term for the BO equation is a singular non-local operator and is *weaker* than the dispersive term found in the KdV equation

$$\partial_t u + \partial_x^3 u + u\partial_x u = 0.$$

The classical problem for (1.0.1) has been concerned with finding minimal regularity in the Sobolev scale

$$H^s(\mathbb{R}) = \left(1 - \partial_x^2\right)^{-s/2} L^2(\mathbb{R}), \quad s \in \mathbb{R}, \tag{1.0.3}$$

for which the IVP (1.0.1) is locally or globally well-posed i.e., precisely that existence and uniqueness hold in a space embedded in $C([0,T] : H^s(\mathbb{R}))$, with the flow map data→solution,

$$H^s(\mathbb{R}) \longrightarrow C([0,T] : H^s(\mathbb{R})),$$

being locally continuous. Molinet-Saut-Tzvetkov showed that for any $s \in \mathbb{R}$ the map data→solution from $H^s(\mathbb{R})$ to $C([0,T] : H^s(\mathbb{R}))$ cannot be locally C^2 in [20]. In particular, this tells us that no local well-posedness result can be obtained solely by applying a contraction principle argument to the equivalent integral formulation of (1.0.1). Namely, as a fixed point of the operator Φ given by

$$\Phi u(x,t) = W(t)u_0(x) - \int_0^t W(t-t')u\partial_x u \; dx.$$

That is, the existence of a fixed point of the operator Φ combined with the Implicit Function Theorem would imply that the map data$\rightarrow$solution is smooth.

Nonetheless, well-posedness results have been established: $s > 3/2$ in [1], [13]; $s \geq 3/2$ in [23]; $s > 5/4$ in [17]; $s > 9/8$ in [16]; $s \geq 1$ in [25]; $s > 1/4$ in [5]; and finally $s \geq 0$ in [12](see also [19] for further results and comments).

Another critical difference between the KdV and the BO is the decay properties of their traveling wave solutions. This will be further investigated in section 2.3.

The scaling analysis for solutions of the BO shows that if $u(x,t)$ is a solution to (1.0.1), then so is $u_\lambda = \lambda u(\lambda x, \lambda^2 t)$ with data $\lambda u_0(\lambda x)$. It is readily verified that

$$\|\lambda u_0(\lambda x)\|_{\dot{H}^s} = \lambda^{s+1/2}\|u_0\|_{\dot{H}^s} \tag{1.0.4}$$

where $\dot{H}^s$ is the homogeneous Sobolev space of order s. When $s = -1/2$, we describe the IVP (1.0.1) as *critical* with respect to the $\dot{H}^s$ norm.

Real valued solutions of the IVP (1.0.1) satisfy infinitely many conservation laws including the following three:

$$\begin{gathered} I_1(u) = \int_{-\infty}^{\infty} u(x,t)dx, \qquad I_2(u) = \int_{-\infty}^{\infty} u^2(x,t)dx, \\ I_3(u) = \int_{-\infty}^{\infty} (|D^{1/2}u|^2 - \frac{u^3}{3})(x,t)dx, \end{gathered} \tag{1.0.5}$$

where $D = \mathcal{H}\,\partial_x$. A discussion of the independence of these quantities from time is found in section 2.2. The conserved quantities allows one to deduce global well-posedness results from local well-posedness results.

Our aim here is to study real valued solutions of the IVP (1.0.1) in weighted Sobolev spaces

$$Z_{s,r} = H^s(\mathbb{R}) \cap L^2(|x|^{2r}dx), \quad s, r \in \mathbb{R}. \tag{1.0.6}$$

In that direction, we have the following theorem:

Theorem A. (Iório 1986/2003 [13], [14]).

1. *If* $r = 1, 2$ *and* $r \leq s$*, then the IVP associated to the BO eq. is globally well posed in* $Z_{s,r}$ *In particular, if* $u_0 \in Z_{s,r}$*, then* $u \in C(\mathbb{R} : Z_{s,r})$.

2. *If* $r = 3$ *and* $3 \leq s$*, then the IVP associated to the BO eq. is globally well posed in* $\dot{Z}_{s,3}$*. In particular, if* $u_0 \in \dot{Z}_{s,3}$*, then* $u \in C(\mathbb{R} : \dot{Z}_{s,3})$*, where* $\dot{Z}_{s,r} = Z_{s,r} \cap \{\widehat{f}(0) = 0\}$.

3. *If* $u \in C(\mathbb{R} : \dot{Z}_{4,3})$ *is a solution of the BO eq. such that*

$$u(\cdot, t_j) \in Z_{4,4}, \quad j = 1, 2, 3,$$

for different times $0 < t_1 < t_2 < t_3$ *then* $u \equiv 0$.

Observe that $\widehat{f}(0)$ propagates with the solution flow according to the first conservation law. Hence the property $\widehat{f}(0) = 0$ is preserved under the solution flow. The result by Iório for integer values was extended to real values of r optimally as we see in the following theorem:

Theorem B. (Fonseca-Ponce 2011 [11]).

1. *If* $r < 5/2$ *and* $r \leq s$*, then the IVP associated to the BO eq. is globally well posed in* $Z_{s,r}$.

2. *If* $5/2 \le r < 7/2$ *and* $r \le s$, *then the IVP associated to the BO eq. is globally well posed in* $\dot{Z}_{s,r}$.

3. *If* $u \in C([0,T] : Z_{s,2})$ *is a solution of the BO eq. such that for* $t_1 \neq t_2 \in [0,1]$ *with* $u(\cdot, t_j) \in Z_{s,5/2}$ *for* $j = 1, 2$, *then*

$$\widehat{u}(0) = \int u(x,t)dx = 0,$$

for all $t \in [0,1]$.

4. *If* $u \in C(\mathbb{R} : \dot{Z}_{7/2,3})$ *is a solution of the BO eq. such that*

$$u(\cdot, t_j) \in Z_{7/2,7/2}, \quad j = 1, 2, 3,$$

for different times $0 < t_1 < t_2 < t_3$ *then* $u \equiv 0$.

In order to obtain a uniqueness property, it is not enough to have a decay requirement of the solution at two different times, namely where the decay parameter is $r = 7/2$. The question arising is *can we have uniqueness at two different times by strengthening the decay assumption?* The following theorem reveals that the condition involving three different times in Theorem A.3 is necessary even by increasing the decay requirement to $r = 4$..

Theorem C. (Fonseca-Linares-Ponce 2012 [10])**.** *For any* $u_0 \in \dot{Z}_{5,4}$ *such that*

$$\int_{\mathbb{R}} x u_0(x)dx \neq 0,$$

the corresponding solution $u \in C(\mathbb{R} : \dot{Z}_{5,7/2^-})$ *of the BO given by Theorem B satisfies that*

$$u(\cdot, t^*) \in \dot{Z}_{4,4},$$

where

$$t^* = -\frac{4}{\|u_0\|_2^2} \int_{\mathbb{R}} x u_0(x) dx.$$

Moreover, in [10] it was shown that if the initial data u_0 is assumed to be in $\dot{Z}_{5,7/2}$ and

$$\int_{\mathbb{R}} x u_0(x) \, dx = 0,$$

then the condition $u(t) \in \dot{Z}_{5,7/2}$ as in Theorem B.3 can be reduced to two times.

1.1 Main Results

In view of the results presented in the previous section, one may expect to have a uniqueness result with a condition at two times in $\dot{Z}_{s,r}$ by increasing the value of r concerning decay to $r > 4$. In this dissertation, we will build on this, and improve the range of the decay parameter to include $r \in (4, 5]$.

Theorem 1. *There exists* $u_0 \in \dot{Z}_{7,5}$ *such that*

$$\int_{\mathbb{R}} x u_0(x) dx \neq 0,$$

for which the corresponding solution $u \in C(\mathbb{R} : \dot{Z}_{7,7/2^-})$ *of the BO satisfies that*

$$u(\cdot, t^*) \in \dot{Z}_{7,5},$$

where

$$t^* = -\frac{4}{\|u_0\|_2^2} \int_{\mathbb{R}} x u_0(x) dx.$$

The existence of data described in Theorem 1 arises from a four parameter family of Schwartz class functions and is discussed in section A. Our main concern is in regards to the maximum space decay properties (r parameter) and not with the minimal regularity needed to obtain it.

Remark 1. The regularity requirement in Theorem 1, namely $s = 7$, is somewhat technical and may be reduced to $s = 6$ with our argument. However, for the sake of simplicity in the proof of Theorems 1 and 2, we keep it as $s = 7$.

We will prove the result in Theorem 1 by considering the time evolution of the moments up to second order of the solutions to (1.0.1). In fact, we seek special cancellation properties that result from the interaction between the linear propagator and the nonlinear integral term. Furthermore, we will show that the condition at three times is necessary even if the value of r is increased to $r \in (5, 5 + 1/2^-]$.

Theorem 2. *There exist infinitely many* $u_0 \in \dot{Z}_{7,5+1/2^-}$ *such that*

$$\int_{\mathbb{R}} x u_0(x) dx \neq 0,$$

for which the corresponding solution $u \in C(\mathbb{R} : \dot{Z}_{7,7/2^-})$ *of the BO satisfies that*

$$u(\cdot, t^*) \in \dot{Z}_{7,5+1/2^-},$$

where t^* *is given in Theorem 1.*

The results in Theorems 1 and 2 are obtained as a consequence of a pure non-linear phenomenon. See Remark 2 for discussion of the corresponding behavior displayed by the linear propagator.

When the associated linear IVP is considered (see section 2.1 for further discussion),

$$\begin{cases} \partial_t v + \mathcal{H}\partial_x^2 v = 0, \quad x, t \in \mathbb{R} \\ v(x,0) = v_0(x) \end{cases} \tag{1.1.1}$$

one writes the solution as

$$v(x,t) = \mathcal{W}(t)v_0(x) = \left(e^{-it\xi|\xi|}\widehat{v_0}\right)^{\check{}}(x).$$

The collection $\{\mathcal{W}(t)\}_{t\in\mathbb{R}}$ defines a *unitary group* of operators in $L^2(\mathbb{R})$. That is, the collection includes an identity $\mathcal{W}(0) = I$, and it is additive in t. Futhermore, the map $t \to H^s$ via $t \to \mathcal{W}(t)v$ is continuous in the L^2 norm for all $u \in L^2$. Note that on $L^2(\mathbb{R})$, $\mathcal{W}(t)$ preserves L^2-norm.

Remark 2. It is easy to see that if $v_0 \in Z_{s,k}$ for $s \geq k$ and $k \in \mathbb{Z}^+$, then $v \in C(\mathbb{R} : Z_{s,k})$ if and only in

$$\int x^j v_0(x)dx = 0 \qquad j = 0, 1 \ldots k-3, \text{ if } k \geq 3. \tag{1.1.2}$$

Notice that if $v_0 \in Z_{k,k}$ for $k \in \mathbb{Z}_+$ and $v(\cdot, t) \in L^2(|x|^{2k}dx)$, then the property in (1.1.2) is preserved by the linear flow.

In the *nonlinear case* the moments are not preserved by the solution flow, thus creating a pure nonlinear phenomenon. Below we state some properties concerning the time evolution of the momentums of the solution whose proof can be found in section 3.1.

Proposition 1. *For $u(x,t)$ a solution of* (1.0.1)

(a)

$$\frac{d}{dt}\int xu(x,t)dx = \frac{1}{2}I_2(u),$$

(b)

$$\frac{d}{dt}\int xu^2(x,t)dx = 2I_3(u),$$

(c)

$$\frac{d}{dt}\int x^2u(x,t)dx = \int xu^2(x,t)dx,$$

where I_2 and I_3 were defined in (1.0.5).

We shall combine these identities with special cancellation properties between the linear and nonlinear terms in the Duhamel formula given in (3.2.2) for solutions of (1.0.1) with a particular class of data to obtain a proof of Theorem 1. Theorem 2 builds on this and an extra condition avoiding the estimate of any additional moments.

Notice that an improvement of our argument shall involve the description of the time evolution of the third moment,

$$\int x^3 u(x,t)dx.$$

However, since the most one can show for a nonzero solution is $u \in C(\mathbb{R} : \dot{Z}_{s,7/2^-})$, then for any $s > 7/2$, our result based on the evolution of the moments up to order two is the best possible. More precisely,

$$\int \frac{\langle x\rangle^{3+1/2^+}}{\langle x\rangle^{1/2^+}}\, udx \leq c\|\langle x\rangle^{7/2^+}u\|_2,$$

and it is unknown if the right hand side is finite.

The rest of this dissertation is organized as follows: Chapter 2 contains an overview of preliminary ideas. Chapter 3 contains the proof of Proposition 1 which is mainly concerned with the justification of the integration by parts (see also Remark 4), and will recall the main argument given in [10] to prove Theorem C. The proofs of Theorems 1 and 2 will also be found in Chapter 3.

Chapter 2

Preliminaries

We introduce terminology and some definitions used throughout this dissertation (see [18, 7]). The broad area of topics discussed in this thesis is that of *nonlinear dispersive partial differential equations.* A system governed by a dispersive partial differential equation (dispersive PDE) is one whose wave solutions spread out in space as they evolve in time. That is, *dispersion* refers to the phenomenon under which waves of different wavelength propagate at different velocities.

Example 1. We illustrate the use of the terminology with the following example. Let us consider the linear KdV:

$$\partial_t u + \partial_x^3 u = 0, \qquad x, t \in \mathbb{R}. \tag{2.0.1}$$

We take a simple wave solution of the form $u(x,t) = Ae^{i(kx-\omega t)}$ and find that the wave solves the linear KdV if and only if $\omega = -k^3$. This is called the *dispersion* *relation* and shows the frequency as a function of the wave number (i.e., the spatial frequency).

Recall the definition of the *Fourier Transform.* The Fourier transform of a function $f \in L^2(\mathbb{R})$ is $\widehat{f}(\xi)$, where

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx, \qquad \xi \in \mathbb{R}.$$

Stated here are some useful facts of the Hilbert Transform (defined in equation (1.0.2) and more details found in Section 2.4):

- $\mathcal{H}$ is an isometry on $L^2(\mathbb{R})$,
- $\mathcal{H}$ is skew-symmetric,
- It is a *singular integral operator* (see Definition 2),
- $\mathcal{H}(fg) = f\mathcal{H}(g) + g\mathcal{H}(f) + \mathcal{H}(\mathcal{H}(f)\mathcal{H}(g))$.

Remark 3. The following fact of the Hilbert transform will be useful in many of the computations found in this work. Namely, if $\int f = 0$ for an appropriate class of functions, that is, $xf, f \in L^2(\mathbb{R})$, then

$$\mathcal{H}(xf) = x\mathcal{H}f.$$

2.1 The Linear BO

This section is dedicated to an overview of the linear BO considered here:

$$\begin{cases} \partial_t v + \partial_x^2 \mathcal{H} v = 0 \quad x, t \in \mathbb{R}, \\ \\ v(x,0) = v_0(x) \end{cases} \tag{2.1.1}$$

Using the Fourier transform in the x variable, we find

$$\widehat{v}(\xi, t) = \widehat{v_0}(\xi) e^{-4\pi^2 i|\xi|\xi t},$$

and so

$$v(x,t) = \left(e^{-4\pi^2 i|\xi|\xi t}\widehat{v_0}(\xi)\right)^{\check{}} = \mathcal{W}(t)v_0(x).$$

Since this thesis is concerned with decay properties of solutions the BO (1.0.1), we will investigate the *regularity properties* of the linear BO in frequency space. That is, if $v_0 \in L^2(|x|^s dx)$ is a well defined solution to (2.1.1), then by Plancherel, $\widehat{v_0}(\xi) \in H^s(\mathbb{R})$. Regularity refers to when we are given $\widehat{v_0}(\xi) \in H^{s_1}(\mathbb{R})$, with $s_1 > s$, then the solution in frequency space $\widehat{v} \in C([0,T]; H^s(\mathbb{R}))$.

Let $c = -4\pi^2 i$. Computing,

$$\partial_\xi \left(e^{ct|\xi|\xi}\widehat{v_0}\right) = e^{ct|\xi|\xi}\left(2ct|\xi|\widehat{v_0} + \partial_\xi \widehat{v_0}\right).$$

Continuing the analogous computations,

$$\partial_\xi^2 \left(e^{ct|\xi|\xi}\widehat{v_0}\right) = e^{ct|\xi|\xi}\left((2ct|\xi|)^2\widehat{v_0} + 2ct\text{sgn}(\xi)\widehat{v_0} + 4ct|\xi|\partial_\xi\widehat{v_0} + \partial_\xi^2\widehat{v_0}\right)$$

$$\partial_\xi^3 \left(e^{ct|\xi|\xi}\widehat{v_0}\right) = e^{ct|\xi|\xi}\left((2ct|\xi|)^3\widehat{v_0} + 12(ct)^2\xi\widehat{v_0} + 12(ct\xi)^2\right.$$
$$\left. + 2ct\delta(\xi)\widehat{v_0} + 6ct\text{sgn}(\xi)\partial_\xi\widehat{v_0} + 6ct|\xi|\partial_\xi^2\widehat{v_0} + \partial_\xi^3\widehat{v_0}\right)$$

It can be observed that when $\widehat{v_0}(\xi = 0) \equiv 0$, then under appropriate assumptions, $\partial_\xi^3 \widehat{v}(\xi, t) \in L^2_\xi(\mathbb{R})$, or using Plancherel's theorem, $v(x,t) \in L^2_x(x^3 dx)$. Using the definition of Fourier Transform, $\widehat{v_0}(\xi = 0) \equiv 0$ means precisely

$$\int v_0 \; dx = 0,$$

also referred as the data, $v_0(x)$, having mean value equal to zero and is naturally a characteristic of functions in the weighted Sobolev spaces we are considering.

Continuing in this fashion, as in (3.2.1), we notice that the next derivative reveals a bad term in the form $\delta \partial_\xi \widehat{v_0}$. Essentially, (more details presented in (3.2.4)), for any hope of having this bad term to be in $L^2_\xi(\mathbb{R})$, we require that

$$\int x v_0 \; dx,$$

vanishes. This is the first moment of the data, and if v_0 is nonzero, there is no guarantee that this term will vanish as it evolves in time for any t. As we shall see in section 3.2, the nonlinear term in the BO provides an opportunity for a good cancellation of this bad term coming from the linear propagator. This idea will reoccur later in Chapter 3.

2.2 Conserved Quantities of solutions to the BO

The proof that the quantities $I_1(u)$ and $I_2(u)$, defined in (1.0.5), do not depend on t will now be presented. Computing the time derivatives, we have

$$\begin{aligned}\frac{d}{dt}\int u(x,t)\ dx = \int \partial_t u\ dx &= -\int \mathcal{H}\partial_x^2 u\ dx - \int u\partial_x u \\ &= \int e^{-2\pi i x\xi}\mathcal{H}\partial_x^2 u\ dx\Big|_{\xi=0} - \int \partial_x\left(\frac{u^2}{2}\right) dx \\ &= \frac{u^2}{2}\Big|_{-\infty}^{\infty} = 0.\end{aligned}$$

The last equality follows from assumptions on u that verify that $u \in H^1$ (see remark 4).

$$\begin{aligned}\frac{d}{dt}\int u^2\ dx &= \int 2\partial_t u \cdot u\ dx \\ &= -2\left(\int u\mathcal{H}\partial_x^2 u\ dx + \int u^2\partial_x u\ dx\right) \\ &= -2\left(u\partial_x u\Big|_{-\infty}^{\infty} - \int \partial_x u\mathcal{H}\partial_x u\ dx + \frac{u^3}{3}\Big|_{-\infty}^{\infty}\right).\end{aligned}$$

The anti-symmetry property belonging to the Hilbert transform was used and the integration-by-parts justified in the spirit of remark 4.

In section 3.1, equation (3.1.7) we utilized the relationship between the second moment of solutions to the Benjamin-Ono equation and the third conserved quantity of solutions to the BO, namely $I_3(u)$, (see 1.0.5). Here we provide the details of the proof that

$$\frac{d}{dt}\int |D^{1/2}u|^2 + \frac{u^3}{3} = 0.$$

We begin the proof by noting the following:

$$\begin{aligned}\frac{d}{dt}|D^{1/2}u|^2 &= \frac{d}{dt}\left[D^{1/2}u\cdot\overline{D^{1/2}u}\right]\\&= \frac{d}{dt}D^1/2u\cdot\overline{D^{1/2}u}+D^{1/2}u\cdot\frac{d}{dt}\overline{D^{1/2}u}\\&= D^{1/2}\partial_t u\cdot\overline{D^{1/2}u}+D^{1/2}u\cdot\overline{D^{1/2}\partial_t u}.\end{aligned}$$

Now, taking $D^{1/2}$ of equation (1.0.1), and then multiplying on both sides by $\overline{D^{1/2}u}$, we have

$$\overline{D^{1/2}u}\cdot D^{1/2}\partial_t u+\overline{D^{1/2}u}\cdot D^{1/2}H\partial_x^2 u+\overline{D^{1/2}u}\cdot D^{1/2}u\partial_x u=0. \tag{2.2.1}$$

After taking $D^{1/2}$ of equation (1.0.1) followed by taking its complex conjugate and finally, multiplying by $D^{1/2}u$, we arrive at the following useful equation

$$D^{1/2}u\cdot\overline{D^{1/2}\partial_t u}+D^{1/2}u\cdot\overline{D^{1/2}H\partial_x^2 u}+D^{1/2}u\cdot\overline{D^{1/2}u\partial_x u}=0. \tag{2.2.2}$$

We now add equations (2.2.1) and (2.2.2) and integrate to obtain:

$$\begin{aligned}\int\frac{d}{dt}|D^{1/2}u|^2dx &= \int\overline{D^{1/2}u}\cdot D^{1/2}\partial_t u+D^{1/2}u\cdot\overline{D^{1/2}\partial_t u}\;dx\\&= -\int\Big[\overline{D^{1/2}u}\cdot D^{1/2}H\partial_x^2u+\overline{D^{1/2}u}\cdot D^{1/2}u\partial_x u\\&\qquad +D^{1/2}u\cdot\overline{D^{1/2}u}\cdot\overline{D^{1/2}u\partial_x u}\Big]\,dx\\&= -\int\Big[\overline{D^{1/2}u}\cdot D^{1/2}H\partial_x^2u+\overline{D^{1/2}u}\cdot D^{1/2}u\partial_x u\\&\qquad +D^{1/2}u\cdot\overline{D^{1/2}u}\cdot\overline{D^{1/2}u\partial_x u}\Big]\,dx\\&= -(I+II+II+VI).\end{aligned}$$

Let us look at term I. Using Parseval's identity we have the following:

$$\begin{aligned}
I = \int \overline{D^{1/2}u} \cdot D^{1/2} H \partial_x^2 u \; dx &= \int \overline{((2\pi|\xi|)^{1/2}\hat{u})\check{}} \left((2\pi|\xi|)^{1/2}\widehat{H\partial_x^2 u}\right)\check{} \; dx \\
&= -i \int \overline{(2\pi|\xi|)^{1/2}\hat{u}} \; (2\pi|\xi|)^{1/2} \mathrm{sgn}(\xi) \widehat{\partial_x^2 u}(\xi) \; d\xi \\
&= -2\pi i \int |\xi| \bar{\hat{u}} \; \mathrm{sgn}(\xi)(2\pi i \xi)^2 \hat{u}(\xi) \; d\xi \\
&= (2\pi)^3 i \int \xi |\xi|^2 \bar{\hat{u}} \hat{u} \; d\xi \\
&= (2\pi)^3 i \int \xi |\xi|^2 |\hat{u}|^2 \; d\xi.
\end{aligned}$$

Similarly, we now look at term II. We have that:

$$\begin{aligned}
II = \int \overline{D^{1/2}u} \cdot D^{1/2}(u\partial_x u) \; dx &= \int \overline{((2\pi|\xi|)^{1/2}\hat{u})\check{}} \left((2\pi|\xi|)^{1/2}\widehat{u\partial_x u}\right)\check{} \; dx \\
&= \int \overline{((2\pi|\xi|)^{1/2}\hat{u})} (2\pi|\xi|)^{1/2} \widehat{u\partial_x u} \; d\xi \\
&= \int \overline{((2\pi|\xi|)^{1/2}\hat{u})} (2\pi|\xi|)^{1/2} \frac{1}{2} \widehat{\partial_x (u^2)} \; d\xi \\
&= 2\pi^2 \int |\xi| \bar{\hat{u}} \cdot (2\pi i \xi) \; \widehat{u^2} \; d\xi \\
&= 2\pi^2 i \int \xi |\xi| \bar{\hat{u}} \widehat{u^2} \; d\xi
\end{aligned}$$

Now, looking at term III, we have:

$$\begin{aligned}
III = \int D^{1/2}u \cdot \overline{D^{1/2} H \partial_x^2 u} \; dx &= \int \left((2\pi|\xi|)^{1/2}\hat{u}\right)\check{} \; \overline{((2\pi|\xi|)^{1/2}\widehat{H\partial_x^2 u})\check{}} \; d\xi \\
&= -\int (2\pi|\xi|)^{1/2}\hat{u} \cdot \overline{(2\pi|\xi|)^{1/2} \; (i \cdot \mathrm{sgn}(\xi) \widehat{\partial_x^2 u})} \; d\xi \\
&= -2\pi \int |\xi| \hat{u} \; \overline{(i \cdot \mathrm{sgn}(\xi)(2\pi i \xi)^2 \hat{u})} \; d\xi \\
&= -(2\pi)^3 i \int \xi |\xi|^2 \hat{u} \bar{\hat{u}} \; d\xi \\
&= -(2\pi)^3 i \int \xi |\xi|^2 |\hat{u}|^2 \; d\xi.
\end{aligned}$$

From term VI, using a change of variables, we can see that:

$$
\begin{aligned}
VI = \int D^{1/2}u \cdot \overline{D^{1/2}(u\partial_x u)}\ dx &= \int \left((2\pi|\xi|)^{1/2}\hat{u}\right)^{\check{}}\ \overline{((2\pi|\xi|)^{1/2}\widehat{u\partial_x u})^{\check{}}}\ dx \\
&= \int (2\pi|\xi|)^{1/2}\hat{u} \cdot \overline{(2\pi|\xi|)^{1/2} \cdot \frac{1}{2}\widehat{\partial_x(u^2)}}\ d\xi \\
&= \pi \int |\xi|\hat{u}\ \overline{(2\pi i\xi)\widehat{u^2}}\ d\xi = -2\pi^2 i \int \xi|\xi|\ \hat{u} \cdot \widehat{u^2}\ d\xi \\
&= -2\pi^2 i \int \xi|\xi|\ \hat{u} \cdot \widehat{\overline{u^2}}\ d\xi \\
&= -2\pi^2 i \int \xi|\xi|\ \hat{u} \cdot \overline{\widehat{u^2}}(-\xi)\ d\xi \\
&= 2\pi^2 i \int \xi|\xi|\ \overline{\hat{\bar{u}}}(-\xi) \cdot \widehat{u^2}(\xi)\ d\xi \\
&= 2\pi^2 i \int \xi|\xi|\ \bar{\hat{u}}(\xi)\widehat{u^2}(\xi)\ d\xi.
\end{aligned}
$$

Thus we have shown that

$$
\int \frac{d}{dt}|D^{1/2}u|^2 dx = -(I + II + III + VI) = -4\pi^2 i \int \xi|\xi|\ \overline{\hat{u}}\widehat{u^2}\ d\xi.
$$

We now turn our attention to the following:

$$
\begin{aligned}
\int \frac{d}{dt}\frac{u^3}{3}\ dx = \int u^2\partial_t u\ dx &= -\int u^2(H\partial_x^2 u + u\partial_x^2 u)\ dx \\
&= -\int u^2 H\partial_x^2 u\ dx - \frac{1}{4}\int \partial_x(u^4)\ dx \\
&= -\int u^2 H\partial_x^2 u\ dx = -\int \bar{\bar{u}}^2 H\partial_x^2 u\ dx \\
&= -\int \bar{\hat{\bar{u}}}^2 \cdot (-i\ \mathrm{sgn}(\xi)\widehat{\partial_x^2 u})\ d\xi \\
&= i\int \bar{\hat{\bar{u}}}^2 \cdot (2\pi i\xi)^2\hat{u}\ d\xi = -(2\pi)^2 i \int \xi|\xi|\ \hat{u}(\xi)\widehat{u^2}(-\xi)\ d\xi \\
&= -4\pi^2 i \int \xi|\xi|\ \bar{\hat{u}}(-\xi)\widehat{u^2}(-\xi)\ d\xi \\
&= 4\pi^2 i \int \xi|\xi|\ \overline{\hat{u}}\widehat{u^2}\ d\xi.
\end{aligned}
$$

Notice that since $u \in H^{1/2+}(\mathbb{R})$, the term

$$\frac{1}{4}\int \partial_x(u^4)\ dx = \frac{u^4}{4}\bigg|_{-\infty}^{\infty} = 0,$$

which was used above. Finally, we are ready to conclude thatt

$$\begin{aligned}\frac{d}{dt}\int\left[|D^{1/2}u|^2 + \frac{u^3}{3}\right]\ dx &= I + II + III + VI + \int \frac{d}{dt}\frac{u^3}{3}\ dx\\ &= -(2\pi)^3 i\int \xi|\xi|^2|\hat{u}|^2\ d\xi - 2\pi^2 i\int \xi|\xi|\bar{\hat{u}}\widehat{u^2}\ d\xi\\ &\quad + (2\pi)^3 i\int \xi|\xi|^2|\hat{u}|^2\ d\xi\\ &\quad - 2\pi^2 i\int \xi|\xi|\ \bar{\hat{u}}(\xi)\widehat{u^2}(\xi)\ d\xi + 4\pi^2 i\int \xi|\xi|\ \bar{\hat{u}}\widehat{u^2}\ d\xi\\ &= 0.\end{aligned}$$

2.3 Dichotomies between traveling waves of the KdV and BO equations

In 1834, Sir John Scott Russell observed a wave traveling across the Union Canal which seemed to maintain its shape and the speed at which it was traveling. The naval architect went home and conducted a series of experiments where he was able to find special relationships between the speed and amplitude of waves traveling at the surface of shallow water in a narrow canal. His findings seemed to contradict water theories known at that time. Then in 1895 Korteweg and de Vries develop the KdV equation which has solitary wave solutions and models wave phenomenon for shallow water in a narrow canal. Recall the IVP for the KdV:

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = 0 \qquad x, t \in \mathbb{R} \\ \\ u(x, 0) = u_0(x). \end{cases} \tag{2.3.1}$$

Before we further our discussion of traveling waves, let us review the scaling argument for solutions of the KdV, (for further comments concerning the BO, see equation (1.0.4)). If $u(x, t)$ is a solution for the KdV, then so is $u_\lambda(x, t) = \lambda^2 u(\lambda x, \lambda^3 t)$. In regards to traveling waves for KdV,

$$u_\lambda(x, t) = \lambda^2 \phi(\lambda(x - \lambda^2 t)) = \phi_\lambda(x - t). \tag{2.3.2}$$

We are looking for $\phi \geq 0$, even, and satisfying the following boundary conditions: $\phi^{(j)}(x) \to 0$ as $|x| \to \infty$ for $j = 0, 1, 2$. Thus we have to solve

$$-\phi + \phi'' + \frac{\phi^2}{2} = 0.$$

One finds that

$$\phi(x) = 3\text{sech}^2\left(\frac{x}{2}\right)$$

is our solution.

We now turn to the Benjamin-Ono equation. If $u(x, t)$ solves the BO then $u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t)$ is also a solution. The traveling waves for BO satisfy:

$$u_\lambda(x, t) = \lambda \psi(\lambda(x - \lambda t)) = \psi_\lambda(x - t).$$

We seek traveling waves $u(x,t) = \psi(x-t)$ satisfying the Benjamin Ono equation, that is,

$$-\psi + \mathcal{H}\psi' + \left(\frac{\psi^2}{2}\right)' = 0.$$

Integrating and applying the Fourier transform, we have that:

$$-\widehat{\psi} + 2\pi|\xi|\widehat{\psi} + \frac{1}{2}\widehat{\psi} * \widehat{\psi} = 0.$$

After some computations, we may conclude

$$\psi(x) \sim -\frac{4}{1+x^2}.$$

We observe that the solitary wave solution is symmetric, has very mild decay at infinity and has been shown to be the unique solitary wave solution [2].

Observing the traveling waves of the KdV and BO, we find that traveling waves for the KdV display strong decay at infinity (spatially), while traveling waves for the BO display mild decay. Both equations take in traveling waves solutions that are *orbitally stable*. To simplify, let us state the definition for the KdV case.

Definition 1. A traveling wave solution ϕ_λ is called *orbitally stable* if given $\epsilon > 0$, there exists $\delta > 0$ such that, if $u \in C([0,\infty) : H^1(\mathbb{R}))$ is a solution with $\|u_0 - \phi_\lambda\|_{H^1} \leq \delta$, for some $\lambda > 0$, then for all $t \in [0,\infty)$, there exists $x(t)$ such that

$$\|u(\cdot + x(t), t) - \phi_\lambda(\cdot)\|_{H^1} < \epsilon.$$

In the case of the BO, the definition is given similarly and H^1 is replaced by $H^{1/2}$. Moreover, as mentioned in Chapter 1 the classical problem (existence, uniqueness, persistence) for the KdV can be solved using an argument based on the contraction mapping principle, whereas the classical problem for the BO is much more involved and cannot be solved solely using a contraction principle argument.

One final remark before ending this section will be made regarding the integrability of both systems. The KdV and BO are known to have an associated *Lax pair*, that is a pair of operators that satisfy a certain commutability requirement, and both equations are known to satisfy infinitely many conserved quantities. They are both solved by inverse scattering transform methods.

2.4 The Hilbert Transform

The Hilbert Transform is a singular integral operator (SIO). It has a special relationship with the Dirichlet problem.

The study of SIOs can be traced back to the 1930s and has since grown into one of the central themes of harmonic analysis.

Definition 2. A singular integral operator in $\mathbb{R}^n$ is a linear operator of the form

$$Lf(x) = \text{p.v.}\frac{\Omega(x)}{|x|^n} * f(x) = \lim_{\epsilon \to 0} \int\limits_{|y|>\epsilon} \frac{\Omega(y')}{|y|^n} f(x-y)\ dy,$$

where $y' = y/|y|$, and

1. Ω is homogeneous of degree zero,

$$\Omega(\lambda x) = \Omega(x) \quad \forall x \in \mathbb{R}^n,\ \forall u > 0,$$

2. Ω is integrable and has mean value zero on the unit sphere in $\mathbb{R}^n$, i.e.,

$$\int_{\mathbb{S}^{n-1}} \Omega(x')d\mathcal{S}_{x'} = 0.$$

In the case where $n = 1$, we notice that the only choice for Ω that has zero average on $\mathbb{S}^0$ is given by $y/|y|$. This corresponds to the Hilbert Transform (1.0.2) and is the only case where $n = 1$.

We now turn our attention to the relationship between the Hilbert transform and the Dirichlet problem, shown here

$$\begin{cases} \partial_x^2 u + \partial_y^2 u = 0 \quad (x, y) \in \mathbb{R} \times (0, \infty), \\ u(x, 0) = f(x) \in L^2(\mathbb{R}). \end{cases} \tag{2.4.1}$$

Taking the Fourier transform in x:

$$\begin{cases} -4\pi^2\xi^2\widehat{u}(\xi, y) + \partial_y^2\widehat{u}(\xi, y) = 0 \\ \widehat{u}(\xi, 0) = \widehat{f}(\xi), \end{cases} \tag{2.4.2}$$

we find that the solution is given by

$$\widehat{u}(\xi, y) = e^{-2\pi|\xi|y}\widehat{f}(\xi) = \frac{1}{\pi}\widehat{\frac{y}{(\cdot)^2 + y^2}}(\xi)\widehat{f}(\xi).$$

The properties of the Fourier transform reveal that the solution in physical space, $u(x, t) = u_f$ is realized as a convolution of the data with the Dirichlet kernel.

Namely,

$$\begin{aligned} u(x,y) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2} f(t)dt \\ &= \frac{1}{\pi}\left(\frac{y}{x^2+y^2} *_x f\right)(x). \end{aligned}$$

Let $z = x+iy$. We have the real and imaginary parts of $\frac{i}{z}$ call them

$$P_y(x) = \frac{y}{x^2+y^2} \qquad Q_y(x) = \frac{x}{x^2+y^2}.$$

Finally,

$$\lim_{y\downarrow 0} \frac{x}{x^2+y^2} = \lim_{y\downarrow 0} Q_y(x) = \text{p.v.}(\frac{1}{x})$$

in the distribution sense.

Let us calculate

$$\frac{1}{\pi}\widehat{\text{p.v.}(\frac{1}{x})}(\xi) \qquad \forall \phi \in \mathcal{S}(\mathbb{R}).$$

We get

$$\begin{aligned} \frac{1}{\pi}\widehat{\text{p.v.}(\frac{1}{x})}\phi &= \frac{1}{\pi}\text{p.v.}(\frac{1}{x})\widehat{\phi} \\ &= \frac{1}{\pi}\lim_{\epsilon\downarrow 0} \int\limits_{\epsilon<|x|<1/\epsilon} \frac{\widehat{\phi}(x)}{x}\, dx \\ &= -i\int_{-\infty}^{\infty} \text{sgn}(y)\phi(y)dy. \end{aligned}$$

For further detail, let the interval $\epsilon < |x| < 1/\epsilon = I$, and let us look closely at

$$\begin{aligned} \lim_{\epsilon\downarrow 0}\int_I \frac{1}{x}\left(\int_{\mathbb{R}} e^{-2\pi iyx}\phi(y)ds\right)dx &= \lim_{\epsilon\downarrow 0}\int_{\mathbb{R}} \phi(y)\left(\int_I \frac{e^{-2\pi iyx}}{x}dx\right)dy \\ &= \int_{\mathbb{R}} \phi(y)\left(\lim_{\epsilon\downarrow 0}\int_I \frac{e^{-2\pi iyx}}{x}dx\right)dy, \end{aligned}$$

where we have used Fubini's theorem. Using the rotational symmetry that the function cosine has about the origin, we see that

$$\begin{aligned}
\lim_{\epsilon\downarrow 0}\int_I \frac{e^{-2\pi ixy}}{x}dx &= \lim_{\epsilon\downarrow 0} -i\int_I \frac{\sin(2\pi xy)}{x}dx \\
&= -2i\int_0^\infty \frac{\sin(2\pi xy)}{x}dx \\
&= -2i\mathrm{sgn}(y)\int_0^\infty \frac{\sin(z)}{z}dz \\
&= -i\pi\mathrm{sgn}(y)
\end{aligned}$$

Above we have used the identity

$$\int_0^\infty \frac{\sin(z)}{z}dz = \frac{\pi}{2}.$$

These computations verify that, as a distribution, the Fourier transform of the principal value of $\frac{1}{x}$ is given by multiplication by $-i\mathrm{sgn}(\cdot)$. Recall the relationship between Fourier transform of convolutions and products, namely $\widehat{g * f} = \widehat{g}\widehat{f}$. Thus

$$\mathcal{H}f(x) = \frac{1}{\pi}\mathrm{p.v.}(\frac{1}{x}) * f(x) = (-i\mathrm{sgn}(\xi)\widehat{f}(\xi))\check{}.$$

Alternatively, we find the analytic extension $F = u_f + iv_f$, where v_f is the complex harmonic conjugate to u_f. The following is an equivalent definition for the Hilbert Transform of the data, f:

$$\lim_{y\downarrow 0} v_f(x,y) = \mathcal{H}f(x).$$

In this context, it is easy to verify the rule for the Hilbert transform of the product of two functions. Given that $\mathcal{H}f$ is the harmonic conjugate of f and $\mathcal{H}g$

is the harmonic conjugate of g, the product of $f + i\mathcal{H}f$ and $g + i\mathcal{H}g$ is analytic, that is

$$(f + i\mathcal{H}f)\,(g + i\mathcal{H}g) = (fg - \mathcal{H}f\mathcal{H}g) + i\,(f\mathcal{H}g + g\mathcal{H}f)\,.$$

This makes $f\mathcal{H}g + g\mathcal{H}f$ the harmonic conjugate of $fg - \mathcal{H}f\mathcal{H}g$, i.e.,

$$f\mathcal{H}g + g\mathcal{H}f = \mathcal{H}(fg) - \mathcal{H}\,(\mathcal{H}f\mathcal{H}g)\,.$$

2.5 Local existence Theory

We will also give an overview of global existence theory using the so-called energy method, although modified versions of the energy method are also used. We begin by stating the following energy estimate as a lemma (see [15], Lemma 2.10):

Lemma 1. *Let $s \geq 0$ and $u \in C([0,T]; H^s(\mathbb{R}))$ be a solution to* (1.0.1). *Then*

$$\sup_{t\in[0,T]} \|u(t)\|_{H^s} \leq c\|u_0\|_{H^s} \exp(cT^{1/2}\|\partial_x u\|_{L^2_T L^\infty_x}). \tag{2.5.1}$$

Existence may follow by a compactness argument using this energy estimate. It is common to use the Gronwall's lemma to find uniqueness of solutions and then a Bona-Smith approximation to verify continuous dependence on the data.

Chapter 3

Proof of Main Results

Proposition 1 provides key descriptions of the time evolution of moments up to second order pertaining to solutions to (1.0.1). In section 2.1 we observed undesired behavior displayed by derivatives in frequency space of the linear propagator. In this chapter, we will investigate the advantages of nonlinear contributions along with the description of a special cancellation in terms of moments up to second order belonging to solutions of the BO equation.

3.1 Proof of Proposition 1

We begin the proof of Proposition 1 by recalling that the proof of (a) is found in (see [13], [14], [21], [10]), and will be included here for completeness. Notice that by multiplying equation (1.0.1) by x and integrating, we arrive at

$$\begin{aligned}\frac{d}{dt}\int xu\;dx &= -\int x\mathcal{H}\partial_x^2 u\;dx - \int xu\partial_x u\;dx\\ &= -\int xu\partial_x u\;dx = -\int x\partial_x\left(\frac{u^2}{2}\right)dx\\ &= -\frac{xu^2}{2}\bigg|_{-\infty}^{\infty} + \frac{1}{2}\int u^2\;dx = \frac{1}{2}I_2(u).\end{aligned}$$

See remark 4 for the justification that the boundary terms from the integration by parts, $\mathcal{H}\partial_x u = xDu$ and xu^2 vanish.

We turn now to the proof of (b). Multiplying the Benjamin-Ono equation (1.0.1) by xu and integrating, we label the three integral terms as follows:

$$\int xu\partial_t u\;dx + \int xu\mathcal{H}\partial_x^2 u\;dx + \int xu^2\partial_x u\;dx = A + B + C = 0.$$

Then

$$A = \frac{1}{2}\frac{d}{dt}\int xu^2\;dx.$$

Also

$$\begin{aligned}B = \int xu\mathcal{H}\partial_x^2 u\;dx = \int xu\partial_x^2\mathcal{H}u\;dx &= xu\partial_x\mathcal{H}u\bigg|_{-\infty}^{\infty} - \int \partial_x(xu)\cdot\partial_x\mathcal{H}u\;dx\\ &= -\int(u + x\partial_x u)\partial_x\mathcal{H}u\;dx\\ &= -\int |D^{1/2}u|^2\;dx - \int x\partial_x u\mathcal{H}\partial_x u\;dx.\end{aligned}$$

Remark 4. Our strategy for justifying the integration by parts performed will often be to use that $H^1(\mathbb{R})$, an algebra, is embedded in $C_\infty(\mathbb{R})$, the space of continuous functions which vanish at infinity. In this chapter, we assume the solution $u(x,t)$ to (1.0.1) is in $C(\mathbb{R} : \dot{Z}_{7,7/2^-})$ as in Theorem B.

Notice that $\partial_x \mathcal{H} u = Du$. Our assumption $u \in C(\mathbb{R} : \dot{Z}_{7,7/2^-})$ implies $Du \in H^6(\mathbb{R}) \subset H^1(\mathbb{R})$. In addition, let us see that $xu \in H^1(\mathbb{R})$, i.e., $xu, \partial_x(xu) \in L^2(\mathbb{R})$. It suffices to check $u, xu, x\partial_x u \in L^2(\mathbb{R})$, which follow from our hypothesis $u \in C(\mathbb{R} : \dot{Z}_{7,7/2^-})$ and Lemma 2 (see Remark 5 below). Thus $xu\partial_x \mathcal{H} u \in H^1(\mathbb{R})$ and

$$xu\partial_x \mathcal{H} u \Big|_{-\infty}^{\infty} = 0.$$

We note here, using the properties of the Hilbert transform (see Remark 3), and the fact that $\widehat{\partial_x u}(\xi = 0) = 0$, that

$$\int x\partial_x u \mathcal{H} \partial_x u \; dx = -\int \partial_x u \mathcal{H}(x\partial_x u) \; dx = -\int x\partial_x u \mathcal{H} \partial_x u \; dx,$$

implying that this term is zero. Hence,

$$B = -\int |D^{1/2} u|^2 dx.$$

It remains to show that $\int u\partial_x \mathcal{H} u \; dx = \int |D^{1/2} u|^2 dx$. By Parseval's theorem, it follows that

$$\begin{aligned}\int u \cdot \mathcal{H}\partial_x u &= \int \bar{\bar{u}} \mathcal{H} \partial_x u = \langle \widehat{\mathcal{H}\partial_x u}, \widehat{\bar{u}} \rangle = -i \int \operatorname{sgn}(\xi)(2\pi i \xi) \hat{u}(\xi) \bar{\hat{u}}(\xi) \; d\xi \\ &= \int (D^{1/2} u)\widehat{} \cdot \overline{(D^{1/2} u)\widehat{}} \; d\xi = \int |D^{1/2} u|^2 \; dx\end{aligned}$$

Similarly, term C becomes

$$C = \int xu^2 \partial_x u \; dx = \frac{1}{3} \int x\partial_x(u^3) dx = \frac{1}{3}[xu^3 \Big|_{-\infty}^{\infty} - \int u^3 dx] = -\int \frac{u^3}{3} dx.$$

As we saw, $xu \in H^1(\mathbb{R})$ by an application of Lemma 2. Our assumptions imply if $u \in H^1(\mathbb{R})$, then $u^2 \in H^1$ (since H^1 is an algebra). Thus, $xu^3 \in H^1(\mathbb{R})$. Hence,

$$A + B + C = \frac{1}{2}\frac{d}{dt}\int xu^2 - \int |D^{1/2}u|^2 - \int \frac{u^3}{3} = 0,$$

and so

$$\frac{1}{2}\frac{d}{dt}\int xu^2 = \int \left(|D^{1/2}u|^2 + \frac{u^3}{3}\right) dx = I_3(u). \tag{3.1.1}$$

This proves Proposition 1 (b).

In the remainder of this dissertation we make use of the following lemma several times (see [4], [21] for a proof)

Lemma 2. *Let $a, b > 0$. Assume that $J^a f = (1-\Delta)^{a/2} f \in L^2(\mathbb{R}^n)$ and $\langle x \rangle^b f = (1 + |x|^2)^{b/2} f \in L^2(\mathbb{R}^n)$. Then for any $\theta \in (0, 1)$*

$$\|\langle x \rangle^{\theta b} J^{(1-\theta)a} f\|_2 \leq c \|\langle x \rangle^b f\|_2^{\theta} \|J^a f\|_2^{(1-\theta)} \tag{3.1.2}$$

and

$$\|J^{(1-\theta)a}\left(\langle x \rangle^{\theta b} f\right)\|_2 \leq c\|\langle x \rangle^b f\|_2^{\theta} \|J^a f\|_2^{1-\theta}. \tag{3.1.3}$$

Remark 5. For $\alpha \in \mathbb{Z}^+$ and $\beta > 0$, we want to bound quantities in the form $\||x|^\beta \partial_x^\alpha u\|_2$ using that $\|\langle x \rangle^{7/2^-} u\|_2$ and $\|J^7 u\|_2$ are finite. Notice that,

$$\begin{aligned} \||x|^\beta \partial_x^\alpha u\|_2 \leq \|\langle x \rangle^\beta \partial_x^\alpha u\|_2 &\leq \|\partial_x^\alpha \left(\langle x \rangle^\beta u\right)\|_2 + \text{ l.o.t.} \\ &\leq \|J^\alpha \langle x \rangle^\beta u\|_2 + \text{ l.o.t.}, \end{aligned}$$

where l.o.t. means lower order terms involving less requirement in the regularity and decay of u and are easier to handle. From Lemma 2, letting $\alpha = a\theta$ and $\beta = b(1-\theta)$, we have

$$\||x|^{\beta}\partial_x^{\alpha}u\|_2 \le \|J^{\alpha}\langle x\rangle^{\beta}u\|_2 + \text{ l.o.t} \le \|J^{a}u\|_2^{\theta}\|\langle x\rangle^{b}u\|_2^{(1-\theta)} + \text{ l.o.t},$$

with $a = 7$ and $b = 7/2^{-}$. We utilize this application of Lemma 2 for $u \in \dot{Z}_{7,7/2^{-}}$ with $\alpha \in \mathbb{Z}^{+}$ and $\beta > 0$, and satisfying the following condition:

$$\frac{\alpha}{7} + \frac{2\beta}{7}^{-} < 1 \tag{3.1.4}$$

to show that terms of the form $\||x|^{\beta}\partial_x^{\alpha}u\|_2$ are finite.

We now recall that $I_3(u)$ is constant in time as it will be useful for proving Proposition 1 part (c), i.e.,

$$\frac{d}{dt}I_3(u)(t) = \frac{d}{dt}\int\left(|D^{1/2}u|^2 + \frac{u^3}{3}\right)dx = 0.$$

The proof of this identity is found in section 2.2. To prove (c) we multiply equation (1.0.1) by x^2 and integrate to obtain

$$\frac{d}{dt}\int x^2u\ dx + \int x^2\mathcal{H}\partial_x^2u\ dx + \frac{1}{2}\int x^2\partial_x(u^2)\ dx = 0. \tag{3.1.5}$$

Notice that

$$\frac{1}{2}\int x^2\partial_x(u^2)\ dx = \frac{1}{2}\left[x^2u^2\Big|_{-\infty}^{\infty} - \int 2xu^2\ dx\right] = -\int xu^2\ dx, \tag{3.1.6}$$

which follows because $xu \in H^1(\mathbb{R})$ thereby justifying the vanishing term from the integration by parts. Also notice that $\int x^2\partial_x^2\mathcal{H}u = 0$ since

$$\begin{aligned}\int x^2\mathcal{H}\partial_x^2 u\ dx = \int x^2\partial_x^2\mathcal{H}u\ dx &= x^2\partial_x\mathcal{H}u\Big|_{-\infty}^{\infty} - \int 2x\partial_x\mathcal{H}u\ dx\\ &= -2\left[x\mathcal{H}u\Big|_{-\infty}^{\infty} - \int \mathcal{H}u\ dx\right] = 2\int \mathcal{H}u\ dx\\ &= 2\widehat{\mathcal{H}u}(0) = -2i\ \mathrm{sgn}(\xi)\widehat{u}(\xi)\big|_{\xi=0} = 0,\end{aligned}$$

This follows because $\widehat{u}(0,t) = \int u(x,t)dx = 0$, a quantity preserved by the solution flow. To justify the integration by parts we shall verify that $x^2\partial_x\mathcal{H}u, x\mathcal{H}u \in H^1(\mathbb{R})$, i.e., $\mathcal{H}u, x\mathcal{H}u, x\partial_x\mathcal{H}u, x^2\partial_x\mathcal{H}u, x^2\partial_x^2\mathcal{H}u \in L^2(\mathbb{R})$. We use that $\widehat{u}(0,t) = 0$ and restrict ourselves to show that $x^2\partial_x^2\mathcal{H}u \in L^2(\mathbb{R})$. But

$$x^2\mathcal{H}\partial_x^2 u = \mathcal{H}(x^2\partial_x^2 u) \in L^2(\mathbb{R}) \Longleftrightarrow x^2\partial_x^2 u \in L^2(\mathbb{R}).$$

Hence the result follows by Remark 5 with $\alpha, \beta = 2$. By equations (3.1.6) and (3.1.5)

$$\frac{d}{dt}\int x^2 u\ dx = \int xu^2\ dx,$$

yielding Proposition 1 (c).

In conjunction with (3.1.1), notice that Proposition 1 (c) implies

$$\frac{d^2}{dt^2}\int x^2 u\ dx = \frac{d}{dt}\int xu^2\ dx = 2I_3(u_0).$$

Thus the second moment of the solution is realized as quadratic in time and takes the form

$$\int x^2 u = t\left[tI_3(u_0) + \int xu_0^2\right] + \int x^2 u_0 = t^2 I_3(u_0) + t\int xu_0^2 + \int x^2 u_0. \quad (3.1.7)$$

3.2 Overview of the Proof of Theorem C

In this section, we recall the main argument in the proof of Theorem C given in [10] presented here for completeness, since we use this notation and need to refer back to these equations. The result is readily motivated by the computations given in section 2.1. Let

$$F_j(t, \xi, u_0) = \partial_\xi^j (e^{-it|\xi|\xi} \hat{u_0}), \quad j = 0, 1, 2, 3, 4, 5.$$

By direct computation,

$$\begin{aligned} F_4(t, \xi, u_0) = \partial_\xi^4 (e^{-it|\xi|\xi} \hat{u_0}) = e^{-it|\xi|\xi} (&16t^4 \xi^4 \hat{u_0} + 48it^3 \xi|\xi| \hat{u_0} - 12t^2 \hat{u_0} \\ &+ 24it^3 |\xi|\xi^2 \partial_\xi \hat{u_0} - 6it\delta \partial_\xi \hat{u_0} - 48t^2 \xi \partial_\xi \hat{u_0} \\ &- 24t^2 \xi^2 \partial_\xi^2 \hat{u_0} - 12it\mathrm{sgn}(\xi) \partial_\xi^2 \hat{u_0} - 8it|\xi| \partial_\xi^3 \hat{u_0} + \partial_\xi^4 \hat{u_0}) \\ = E_1(t, \xi, u_0) + \ldots + E_{10}(t, \xi, u_0).& \end{aligned} \tag{3.2.1}$$

Above we have taken into consideration that the initial data is assumed to have mean value zero.

By Duhamel's Principle, we have that

$$\hat{u}(\xi, t) = F_0(t, \xi, u_0) - \int_0^t F_0(t - t', \xi, u\partial_x u(\xi, t'))dt'. \tag{3.2.2}$$

Thus

$$\partial_\xi^4 \hat{u}(\xi, t) = F_4(t, \xi, u_0) - \int_0^t F_4(t - t', \xi, u\partial_x u(\xi, t'))dt'.$$

Substituting equation (3.2.1) above, and using an argument similar to the one to be provided in the proof of Theorem 1 in section 3.3, we have that all terms of $\partial_\xi^4 \hat{u}(\xi)$ are in $L^2(\mathbb{R})$ except for

$$\Psi_1 = E_5(t,\xi,u_0) - \int_0^t E_5(t-t',\xi,u\partial_x u(\xi,t'))dt'. \tag{3.2.3}$$

Let us inspect $E_5(t,\xi,u_0)$ by adding and subtracting appropriate terms:

$$\begin{aligned} E_5(t,\xi,u_0) &= -6it\delta(\xi)e^{-it|\xi|\xi}\partial_\xi\hat{u_0}(\xi) \qquad (3.2.4) \\ &= -6it\delta(\xi)\left(e^{-it|\xi|\xi}\partial_\xi\hat{u_0}(\xi) - \partial_\xi\hat{u_0}(0)\right) - 6it\delta(\xi)\partial_\xi\hat{u_0}(0) \\ &= -6it\delta(\xi)\partial_\xi\hat{u_0}(0) \\ &= -6it\delta(\xi)\cdot\partial_\xi \int e^{-ix\xi}u_0(x)dx\Big|_{\xi=0} = -6t\delta(\xi)\int xu_0(x)dx \end{aligned}$$

Similarly,

$$\int_0^t E_5(t-t',\xi,u\partial_x u(\xi,t'))dt' = -6i\delta(\xi)\int_0^t (t-t')\partial_\xi\widehat{u\partial_x u}(0,t')dt'. \tag{3.2.5}$$

Notice that by Proposition 1 and the conservation laws,

$$\begin{aligned} \partial_\xi\widehat{u\partial_x u}(0,t') &= \int (-ix)e^{-ix\xi}u\partial_x u(x,t')\ dx\Big|_{\xi=0} \\ &= -i\int xu\partial_x u\ dx = -i\int x\partial_x(\frac{u^2}{2})\ dx \\ &= -i\left[\frac{xu^2}{2}\Big|_{-\infty}^{\infty} - \int\frac{u^2}{2}\right] \\ &= i\int\frac{u^2}{2}\ dx = \frac{i}{2}\|u_0\|_2^2 = i\frac{d}{dt'}\int xu(x,t')\ dx. \end{aligned} \tag{3.2.6}$$

Above we used that $xu^2 \in H^1(\mathbb{R})$ to justify the integration by parts, as we have done previously. We substitute (3.2.6) into (3.2.5) to obtain, using integration by parts,

$$\begin{aligned}
\int_0^t E_5(t-t',\xi,u\partial_x u(\xi,t'))dt' &= -6i\int_0^t (t-t')\delta(\xi)\partial_\xi\widehat{u\partial_x u}(0,t')dt' \\
&= 6\delta(\xi)\int_0^t (t-t')\left\{\frac{d}{dt'}\int xu(x,t')dx\right\}dt' \\
&= 6\delta(\xi)\left[(t-t')\int_{\mathbb{R}} xu(x,t')dx\Big|_{t'=0}^{t'=t} + \int_0^t\int_{\mathbb{R}} xu(x,t')dx\ dt'\right] \\
&= -6t\delta(\xi)\int xu_0(x)dx + 6\delta(\xi)\int_0^t\left(\int xu(x,t')dx\right)dt'.
\end{aligned}$$

The equation above together with equations (3.2.3) and (3.2.4) tell us that

$$\Psi_1 = -6\delta(\xi)\int_0^t\left(\int xu(x,t')dx\right)dt'. \tag{3.2.7}$$

Notice above that δ depends on ξ and we have removed the dependence on ξ from the rest of Ψ_1. That is, there is an opportunity to discover some special cancellation property.

Recall that from equation (3.2.6),

$$\frac{d}{dt'}\int xu(x,t')dx = \frac{1}{2}\|u_0\|_2^2.$$

Hence

$$\int_{\mathbb{R}} xu(x,t')dx = \frac{t'}{2}\|u_0\|_2^2 + \int xu_0(x)dx.$$

Thus doing the integration in the variable t' we rewrite equation (3.2.7) as

$$\begin{aligned}
\Psi_1 = -6\delta(\xi)\int_0^t\left(\int xu(x,t')dx\right)dt' &= -6\delta(\xi)\int_0^t\left(\frac{t'}{2}\|u_0\|_2^2 + \int xu_0(x)dx.\right)dt' \\
= -6\delta(\xi)\left[\frac{t^2}{4}\|u_0\|_2^2 + t\int xu_0(x)dx\right] &= -6t\delta(\xi)\left[\frac{t}{4}\|u_0\|_2^2 + \int xu_0(x)dx\right].
\end{aligned}$$

Define Φ_1 as

$$\Phi_1 = \frac{t^*}{4}\|u_0\|_2^2 + \int xu_0(x)dx. \tag{3.2.8}$$

Since $\int xu_0(x)dx \neq 0$, the expression (3.2.3) vanishes when $\Phi_1 = 0$ at time $t^* \neq 0$ where

$$t^* = -\frac{4}{\|u_0\|_2^2}\int xu_0(x)dx,$$

as desired.

3.3 Proof of Theorem 1

In the previous section, we discovered a specific time, t^*, where there is a special cancellation property in Ψ_1 given by equation (3.2.7). Continuing under the assumptions that guarantee the existence of this phenomenon, we are interested in taking the fifth derivative of $\widehat{u}(\xi, t)$. These results are summarized in [8].

We are investigating whether or not the special cancellation property discovered in the previous section for F_4 will somehow carry over to F_5. We begin by computing,

$$
\begin{aligned}
F_5(t,\xi,u_0) = \partial_\xi^5(e^{-it|\xi|\xi}\widehat{u_0}(\xi)) = e^{-it\xi|\xi|}\big(&-32it^5|\xi|\xi^4\hat{u}_0 + 140t^4\xi^3\hat{u}_0 \\
&+ 100it^3|\xi|\hat{u}_0 + 64t^4\xi^4\partial_\xi\hat{u}_0 + 196it^3|\xi|\xi\partial_\xi\hat{u}_0 - 60t^2\partial_\xi\hat{u}_0 \\
&+ 72it^3|\xi|\xi^2\partial_\xi^2\hat{u}_0 - 144t^2\xi\partial_\xi^2\hat{u}_0 - 12it\delta\partial_\xi^2\hat{u}_0 \\
&-40t^2\xi^2\partial_\xi^3\hat{u}_0 - 20it\mathrm{sgn}\xi\partial_\xi^3\hat{u}_0 - 10it|\xi|\partial_\xi^4\hat{u}_0 + \partial_\xi^5\hat{u}_0\big) \\
= E_1(t,\xi,u_0) + \ldots E_{13}(t,\xi,u_0). \qquad (3.3.1)
\end{aligned}
$$

We are now ready to investigate the behaviors or E_i and use the interpolation Lemma 2 to prove the L^2 boundedness of these functions. In fact, we will find below that the L^2 norms of the quantities in (3.3.3) are finite following from the decay and regularity assumptions on the data and Lemma 2. The main idea is summarized in Remark 6 below. Recall that $u_0 \in \dot{Z}_{7,5}$.

Remark 6. In (3.3.3) we are interested in quantities of the form $\||x|^\beta\partial_x^\alpha u_0\|_2$ for $\alpha \in \mathbb{Z}^+$ and $\beta > 0$. In the spirit of Remark 5, using that $u_0 \in \dot{Z}_{7,5}$ if

$$\frac{\alpha}{7} + \frac{\beta}{5} < 1 \qquad (3.3.2)$$

then $\||x|^\beta\partial_x^\alpha u_0\|_2$ can be bounded in terms of $\|J^7u_0\|_2$ and $\|\langle x\rangle^5u_0\|_2$.

Note that $E_i(t,\xi,u_0)$ for $i \neq 9$ is in $L^2(\mathbb{R})$ as desired using the comments from Remark 6:

$$\left\{\begin{aligned}
\|E_1\|_2 \quad &= \|32it^5\,|\xi|\xi^4 e^{-it|\xi|\xi}\widehat{u}_0\|_2 \le c_t\,\|\partial_x^5 u_0\|_2,\\
\|E_2\|_2 \quad &= \|140t^4\,\xi^3 e^{-it|\xi|\xi}\widehat{u}_0\|_2 \le c_t\,\|\partial_x^3 u_0\|_2,\\
\|E_3\|_2 \quad &= \|100it^3\,|\xi| e^{-it|\xi|\xi}\widehat{u}_0\|_2 \le c_t\,\|\partial_x u_0\|_2,\\
\|E_4\|_2 \quad &= \|64t^4\,\xi^4 e^{-it|\xi|\xi}\partial_\xi\widehat{u}_0\|_2 \le c_t\,(\|\partial_x^3 u_0\|_2 + \|x\partial_x^4 u_0\|_2),\\
\|E_5\|_2 \quad &= \|196it^3\,|\xi|\xi e^{-it|\xi|\xi}\partial_\xi\widehat{u}_0\|_2 \le c_t\,(\|\partial_x u_0\|_2 + \|x\partial_x^2 u_0\|_2),\\
\|E_6\|_2 \quad &= \|60t^2 e^{-it|\xi|\xi}\partial_\xi\widehat{u}_0\|_2 \le c_t\,\|xu_0\|_2,\\
\|E_7\|_2 \quad &= \|72it^3\,|\xi|\xi^2 e^{-it|\xi|\xi}\partial_\xi^2\widehat{u}_0\|_2\\
&\qquad \le c_t\,(\|\partial_x u_0\|_2 + \|x\partial_x^2 u_0\|_2 + \|x^2\partial_x^2 u_0\|_2 + \|x\partial_x u_0\|_2),\\
\|E_8\|_2 \quad &= \|144t^2\,\xi e^{-it|\xi|\xi}\partial_\xi^2\widehat{u}_0\|_2 \le c_t\,(\|xu_0\|_2 + \|x^2\partial_x u_0\|_2),\\
\|E_{10}\|_2 \quad &= \|40t^2\,\xi^2 e^{-it|\xi|\xi}\partial_\xi^3\widehat{u}_0\|_2\\
&\qquad \le c_t\,(\|xu_0\|_2 + \|x^2\partial_x u_0\|_2 + \|x^2\partial_x u_0\|_2 + \|x^3\partial_x^2 u_0\|_2),\\
\|E_{11}\|_2 \quad &= \|20it\,\mathrm{sgn}(\xi) e^{-it|\xi|\xi}\partial_\xi^3\widehat{u}_0\|_2 \le c_t\,\|x^3 u_0\|_2,\\
\|E_{12}\|_2 \quad &= \|10it\,|\xi| e^{-it|\xi|\xi}\partial_\xi^4\widehat{u}_0\|_2 \le c_t\,(\|x^3 u_0\|_2 + \|x^4\partial_x u_0\|_2),\\
\|E_{13}\|_2 \quad &= \|e^{-it|\xi|\xi}\partial_\xi^5\widehat{u}_0\|_2 \le c_t\|x^5 u_0\|_2.
\end{aligned}\right. \tag{3.3.3}$$

As in the previous section, by Duhamel's Principle (3.2.2), we have that

$$\partial_\xi^5\hat{u}(\xi,t) = F_5(t,\xi,u_0) - \int_0^t F_5(t-t',\xi,u\partial_x u(\xi,t'))dt'.$$

To illustrate the argument, let us apply the Remark 6 above to $\|x\partial_x^4 u_0\|_2$ from $\|E_4\|_2$. One takes $\alpha = 4$ and $\beta = 1$ and condition (3.3.2) is satisfied proving that $\|x\partial_x^4 u_0\|_2 < \infty$. The same argument applies to show all cases in equation (3.3.3) are finite.

Remark 7. Notice that the decay and regularity of the data, $u_0(x)$ was the key to showing that E_i for $i \neq 9$ are in $L^2(\mathbb{R})$. Furthermore, the boundedness of the terms in (3.3.3) can be also obtained if $u_0 \in \dot{Z}_{6,6}$.

Recall that the assumption $u_0 \in \dot{Z}_{7,5}$ and the result in [11] guarantees that the solution $u \in C([0,T] : \dot{Z}_{7,7/2^-})$.

Claim 1. *If $u \in C([0,T] : \dot{Z}_{7,7/2^-})$, then $u\partial_x u \in C([0,T] : \dot{Z}_{6,6^-})$.*

Proof of Claim 1: Consider $r = r_1 + r_2 > 0$,

$$\int |x|^{2r} |u\partial_x u|^2 dx = \||x|^r u\partial_x u\|_2^2$$

and

$$\||x|^r u\partial_x u\|_2 \leq \||x|^{r_1} u\|_2 \ \||x|^{r_2} \partial_x u\|_\infty \leq \||x|^{r_1} u\|_2 \ \||x|^{r_2} \partial_x u\|_{1,2},$$

where $\|f\|_{1,2} = \|f\|_2 + \|\partial_x f\|_2$. We would like to show that the parameter $r = 6^-$. Notice that we may use $r_1 = \frac{7}{2}^-$. The term requiring the most decay and regularity will be $\||x|^{r_2} \partial_x^2 u\|_2$. Using Remark 5 and condition (3.1.4) with $\alpha = 2$ and $\beta = r_2$, we have that $r_2 = \frac{5}{2}^-$, thus verifying the claim to be true.

Now we can apply the same techniques to prove the boundedness of corresponding integral terms in (3.3.3). We need to direct our attention to the following portion of $\partial_\xi^5 \hat{u}(\xi, t)$ which we will call Ψ_2:

$$\Psi_2 = E_9(t, \xi, u_0) - \int_0^t E_9(t - t', \xi, u\partial_x u(\xi, t'))dt'. \tag{3.3.4}$$

We are interested in proving that $\Psi_2 \in L^2(\mathbb{R})$ for some time $t = t^{**} \neq 0$ by showing that it vanishes at this specific time.

We compute by adding and subtracting appropriate terms to eliminate the exponential term and the dependence of $\partial_\xi^2 \hat{u_0}$ on ξ in the first part of equation (3.3.4),

$$\begin{aligned} E_9(t, \xi, \hat{u_0}) &= e^{-it\xi|\xi|} \left(-12it\delta(\xi)\partial_\xi^2 \hat{u_0}(\xi) \right) \\ &= -12it\delta(\xi) \left(e^{-it|\xi|\xi} \partial_\xi^2 \hat{u_0}(\xi) - \partial_\xi^2 \hat{u_0}(0) \right) - 12it\delta(\xi)\partial_\xi^2 \hat{u_0}(0) \\ &= -12it\delta(\xi)\partial_\xi^2 \hat{u_0}(0) = -12it\delta(\xi)\partial_\xi^2 \int e^{-ix\xi} u_0(x)dx \Big|_{\xi=0} \\ &= 12it\delta(\xi) \int x^2 u_0(x)dx. \end{aligned} \tag{3.3.5}$$

For the integral term of equation (3.3.4) we will use an integration by parts as before

$$\begin{aligned} \partial_\xi^2 \widehat{u\partial_x u}(0) &= -\int x^2 \partial_x \left(\frac{u^2}{2} \right) dx = -\left[\frac{x^2 u^2}{2} \Big|_{-\infty}^{\infty} - \int 2x \cdot \frac{u^2}{2} dx \right] \\ &= \int x u^2 dx = \frac{d}{dt'} \int x^2 u(x, t')dx. \end{aligned}$$

Thus, we compute using the above information, integration by parts in the variable t' and equation (3.3.5),

$$\begin{aligned}
\int_0^t E_9(t-t',\xi,u\partial_x u(\xi,t'))dt' &= -12i\delta(\xi)\int_0^t (t-t')\partial_\xi^2 \widehat{u\partial_x u}(0,t')dt' \\
&= -12i\delta(\xi)\int_0^t (t-t')\cdot\frac{d}{dt'}\int x^2 u(x,t')dx\ dt' \\
&= -12i\delta(\xi)\left[(t-t')\int x^2 u(x,t')dx\Big|_{t'=0}^{t'=t} + \int_0^t\int_{\mathbb{R}} x^2 u(x,t')dx\ dt'\right] \\
&= 12it\delta(\xi)\int_{\mathbb{R}} x^2 u_0(x)dx - 12i\delta(\xi)\int_0^t\int_{\mathbb{R}} x^2 u(x,t')dxdt'
\end{aligned}$$

Recall from equation (3.1.7) that

$$\int x^2 u(x,t)dx = t^2 I_3(u_0) + t\int x u_0^2(x)dx + \int x^2 u_0(x)dx.$$

Hence we see that equation (3.3.4) becomes

$$\begin{aligned}
\Psi_2 &= 12it\delta(\xi)\int_0^t\int x^2 u(x,t')dxdt' \\
&= 12it\delta(\xi)\int_0^t \left\{t'^2 I_3(u_0) + t'\int x u_0^2(x)dx + \int x^2 u_0(x)dx\right\} dt' \\
&= it\delta(\xi)\left[4t^3 I_3(u_0) + 6t^2\int x u_0^2(x)dx + 12t\int x^2 u_0(x)dx\right] \\
&= it^2\delta(\xi)\left[4t^2 I_3(u_0) + 6t\int x u_0^2(x)dx + 12\int x^2 u_0(x)dx\right].
\end{aligned}$$

We label

$$\Phi_2 = 4t^2 I_3(u_0) + 6t\int x u_0^2(x)dx + 12\int x^2 u_0(x)dx. \tag{3.3.6}$$

That is, $\Psi_2 = it^2\delta(\xi)\Phi_2$. Notice that the behavior displayed for Ψ_1 is somewhat generalized here; that is we have Ψ_2 is realized by the product of $\delta(\xi)$ and another term that doesn't depend on ξ, presenting us with an opportunity for a good cancellation property.

It remains to show that there exists a time $t = t^{**} \neq 0$ such that Ψ_2 given by (3.3.4) vanishes and furthermore, that for a class of data $u_0 \in \dot{Z}_{7,5}$, $t^{**} = t^*$, where

$$t^* = -\frac{4}{\|u_0\|_2^2} \int x u_0(x) dx,$$

as in Theorem C whose proof was sketched in the previous section. This will be a consequence of the Implicit Function Theorem and the proof will be given in the appendix (see section A). □

3.4 Proof of Theorem 2

Using the properties given by Plancherel's theorem, we will estimate

$$\|D_\xi^{1/2^-} F_5(t, \xi, u_0)\|_2$$

and the corresponding Duhamel integrals. We use an equivalent formulation of fractional derivatives presented in [24].

Theorem D. (Stein [24]). *Let $b \in (0,1)$ and $2n/(n+2b) < p < \infty$. Then $f \in L_b^p(\mathbb{R}^n)$ if and only if*

(a) $f \in L^p(\mathbb{R}^n)$,

(b)

$$\mathcal{D}^b f(x) = \left(\int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x-y|^{n+2b}} dy \right)^{1/2} \in L^p(\mathbb{R}^n)$$

with

$$\|f\|_{b,p} = \|J^b f\|_p \cong \|f\|_p + \|D^b f\|_p \cong \|f\|_p + \|\mathcal{D}^b f\|_p.$$

For $p = 2$ and $b \in (0,1)$, one can deduce that $\|\mathcal{D}^b f\|_2 = c_n \|D^b f\|_2$ and

$$\|\mathcal{D}^b(fg)\|_2 \leq \|f\mathcal{D}^b g\|_2 + \|g\mathcal{D}^b f\|_2. \tag{3.4.1}$$

Next we recall the following result found in [21].

Proposition 2. *Let $b \in (0,1)$ and $t > 0$. Then $\mathcal{D}^b(e^{-it\xi|\xi|}) \leq c(|t|^{b/2} + |t|^b|\xi|^b)$.*

Proposition 3.

(a) Suppose $f \in H^s(\mathbb{R})$ where $s > \frac{1}{2}$ and $f(0) \neq 0$. Then sgn$(x)f \in H^b(\mathbb{R})$ if and only if $b < \frac{1}{2}$.

(b) Suppose $f \in H^s(\mathbb{R})$ where $0 < s \leq \frac{1}{2}$. If $b < s$, then

$$sgn(x)f \in H^b(\mathbb{R}).$$

Proof. (a) Let $\phi \in C_0^\infty(\mathbb{R}),\ \mathrm{supp}(\phi) \in [-1,1]$ and $\phi(x) \equiv 1$ for $x \in [-1/2, 1/2]$. Thus, $\mathrm{sgn}(x)f \in H^b(\mathbb{R})$ if and only if $\mathrm{sgn}(x)\phi(x)f \equiv \widetilde{\phi}f \in H^b(\mathbb{R})$. We may apply Theorem D to reduce our problem to showing that

$$\mathcal{D}^b(\widetilde{\phi}(x)f(x)) = \left(\int\limits_{\mathbb{R}} \frac{|\widetilde{\phi}(x)f(x) - \widetilde{\phi}(y)f(y)|^2}{|x-y|^{1+2b}} dy \right)^{1/2} \in L^2(\mathbb{R}).$$

Using the properties of $\widetilde{\phi}$ and Fubini's theorem, if suffices to show

$$\int_{-1}^{0}\int_{0}^{1}\frac{|\widetilde{\phi}(x)f(x)-\widetilde{\phi}(y)f(y)|^2}{|x-y|^{1+2b}}dxdy<\infty.$$

We observe that the interesting case occurs for x and y values near the origin and for which the difference in the sign of f matters, i.e., for those $x\in(0,\delta)$ and $y\in(-\delta,0)$ where $|\phi(x)f(x)-\phi(y)f(y)|\geq\lambda$ where δ is small and $\lambda\neq 0$ for almost every appropriate x and y.Without loss of generality, we assume that $\exists\delta\in(0,1)$ such that if $x\in(0,\delta)$ and $y\in(-\delta,0)$ then $|\widetilde{\phi}(x)f(x)-\widetilde{\phi}(y)f(y)|\geq\lambda$ is small and $\lambda\neq 0$.

So it suffices to show that

$$\int_{-\delta}^{0}\int_{0}^{\delta}\frac{dxdy}{|x-y|^{1+2b}}<\infty. \tag{3.4.2}$$

Changing variables, $x-y=h$ and $x=x'$,(3.4.2) is equivalent to

$$\int_{0}^{\delta}\int_{x}^{x+\delta}\frac{dhdx}{h^{1+2b}}=-c\int_{0}^{\delta}h^{-2b}\Big|_{x}^{x+\delta}dx=\int_{0}^{\delta}\left(\frac{c}{x^{2b}}-\frac{c}{(x+\delta)^{2b}}\right)dx$$

where c is a constant that depends on b. Certainly, this is equivalent to

$$c\int_{0}^{\delta}\frac{dx}{x^{2b}}<\infty,$$

if and only if $2b<1$, that is $b<\frac{1}{2}$.

(b) It suffices to consider $\chi_{\mathbb{R}^+}f$. Using a change of variables as before, we have

$$\int_{-1}^{0}\int_{0}^{1}\frac{|f(x)|^2}{|x-y|^{1+2b}}dxdy\cong\int_{0}^{1}\left(\int_{x}^{x+1}\frac{dh}{h^{1+2b}}\right)|f(x)|^2\,dx\cong\int_{0}^{1}\frac{|f(x)|^2}{|x|^{2b}}dx.$$

An application of the generalized Hölder's inequality with $1/2 = 1/p + 1/q$ implies

$$\left\| \frac{f}{|x|^b} \right\|_2^2 \leq \|f\|_p^2 \left\| \frac{\chi_{(0,1)}}{|x|^b} \right\|_q^2.$$

It suffices to have:

1. $H^s \hookrightarrow L^p \Longleftrightarrow s > 1/2 - 1/p$,

2. $bq < 1 \Longleftrightarrow b < 1/q = s$.

$\square$

Recall that $u_0 \in \dot{Z}_{7,11/2^-}$ and that $F_5(t, \xi, u_0) = \sum_{j=1}^{13} E_j(t, \xi, u_0)$ where E_j were defined in (3.3.1). With $b = 1/2^-$ we want to prove that $\|\mathcal{D}^b F_5(t, \xi, u_0)\|_2$ is finite. Applying (3.4.1) and Proposition 2 consider $\|\mathcal{D}^b E_1\|_2$:

$$\left\|\mathcal{D}^b E_1\right\|_2 = c \left\|\mathcal{D}_\xi^b \left(e^{-it|\xi|\xi} |\xi| \xi^4 \widehat{u_0}(\xi)\right)\right\|_2 = c \left\|\mathcal{D}^b \left(e^{-it\xi|\xi|} (\operatorname{sgn}(\xi) \widehat{\partial_x^5 u_0}(\xi))\right)\right\|_2$$

$$\lesssim \left\|\operatorname{sgn}(\xi) \widehat{\partial_x^5 u_0}(\xi) \mathcal{D}^b \left(e^{-it\xi|\xi|}\right)\right\|_2 + \left\|e^{-it\xi|\xi|} \mathcal{D}^b \left((\operatorname{sgn}(\xi) \widehat{\partial_x^5 u_0}(\xi)\right)\right\|_2$$

$$\lesssim \left\{ \left\| |\xi|^b \operatorname{sgn}(\xi) \widehat{\partial_x^5 u_0} \right\|_2 + \left\| \mathcal{D}^b (\operatorname{sgn}(\xi) \widehat{\partial_x^5 u_0}) \right\|_2 \right\}$$

where c may denote different constants and $\lesssim$ means less than or equal to up to a constant which, here, might depend on t. Proposition 3 implies that if $\widehat{\partial_x^5 u_0} \in H^s(\mathbb{R})$ for $s = 1/2$ then $\operatorname{sgn}(\xi) \widehat{\partial_x^5 u_0} \in H^b(\mathbb{R})$ with $b = 1/2^-$. Using the Fourier transform, our task is to show $\||x|^{1/2} \partial_x^5 u_0\|_2 < \infty$. After that, it remains to prove that $\||\xi|^{b+5} \widehat{u_0}\|_2 < \infty$, so we can conclude that $\|\mathcal{D}^b E_1\|_2 < \infty$. Both quantities involved in bounding $\|\mathcal{D}^b E_1\|_2$ are finite by conditions (3.4.4) and (3.4.5), given below. We compute similarly:

$$\begin{cases} \|\mathcal{D}^b E_1\|_2 & \lesssim \||\xi|^{b+5}\widehat{u_0}\|_2 + \|\mathcal{D}^b(\mathrm{sgn}(\xi)\widehat{\partial_x^5 u_0})\|_2, \\ \|\mathcal{D}^b E_2\|_2 & \lesssim \||\xi|^{b+3}\widehat{u_0}\|_2 + \||x|^b\partial_x^3 u_0\|_2, \\ \|\mathcal{D}^b E_3\|_2 & \lesssim \||\xi|^{b+1}\widehat{u_0}\|_2 + \|\mathcal{D}^b(\mathrm{sgn}(\xi)\widehat{\partial_x u_0})\|_2, \\ \|\mathcal{D}^b E_4\|_2 & \lesssim \||\xi|^{b+4}\partial_\xi\widehat{u_0}\|_2 + \||x|^{b+1}\partial_x^4 u_0\|_2, \\ \|\mathcal{D}^b E_5\|_2 & \lesssim \||\xi|^{b+2}\partial_\xi\widehat{u_0}\|_2 + \|\mathcal{D}^b(\mathrm{sgn}(\xi)\widehat{\partial_x^2(xu_0)})\|_2, \\ \|\mathcal{D}^b E_6\|_2 & \lesssim \||\xi|^{b}\partial_\xi\widehat{u_0}\|_2 + \||x|^{b+1}u_0\|_2, \\ \|\mathcal{D}^b E_7\|_2 & \lesssim \||\xi|^{b+3}\partial_\xi^3\widehat{u_0}\|_2 + \|\mathcal{D}^b(\mathrm{sgn}(\xi)\widehat{\partial_x^3(x^2u_0)})\|_2, \\ \|\mathcal{D}^b E_8\|_2 & \lesssim \||\xi|^{b+1}\partial_\xi^2\widehat{u_0}\|_2 + \||x|^{b+2}\partial_x u_0\|_2, \\ \|\mathcal{D}^b E_{10}\|_2 & \lesssim \||\xi|^{b+2}\partial_\xi^3\widehat{u_0}\|_2 + \||x|^{b+3}\partial_x u_0\|_2, \\ \|\mathcal{D}^b E_{11}\|_2 & \lesssim \||\xi|^{b}\partial_\xi^3\widehat{u_0}\|_2 + \|\mathcal{D}^b(\mathrm{sgn}(\xi)\widehat{x^3u_0})\|_2, \\ \|\mathcal{D}^b E_{12}\|_2 & \lesssim \||\xi|^{b+1}\partial_\xi^4\widehat{u_0}\|_2 + \|\mathcal{D}^b(\mathrm{sgn}(\xi)\widehat{\partial_x(x^4u_0)})\|_2, \\ \|\mathcal{D}^b E_{13}\|_2 & \lesssim \||\xi|^{b}\partial_\xi^5\widehat{u_0}\|_2 + \||x|^{b+5}u_0\|_2. \end{cases} \tag{3.4.3}$$

As in Remarks 5 and 6, an application of Lemma 2 tells us $\||x|^\beta\partial_x^\alpha u_0\|_2$ is bounded if

$$\frac{\alpha}{7} + \frac{2\beta^-}{11} < 1, \tag{3.4.4}$$

In addition to condition (3.4.4), we use that $\widehat{u_0} \in Z_{11/2^-,7}$ whenever $\nu \in \mathbb{Z}^+, \eta > 0$, yielding a condition for $\||\xi|^\eta \partial_\xi^\nu \widehat{u_0}\|_2 < \infty$, namely,

$$\frac{2\nu^-}{11} + \frac{\eta}{7} < 1. \tag{3.4.5}$$

The term $\|\mathcal{D}^b E_9\|_2$ is omitted in (3.4.3) since we are in the hypothesis of Theorem 1, so there is cancellation with the Duhamel integral term at time $t = t^*$. Below we consider some terms in (3.4.3), for instance $\|\mathcal{D}^b E_4\|_2$ and $\|\mathcal{D}^b E_{12}\|_2$. For $\|\mathcal{D}^b E_4\|_2$, the term $\||\xi|^{b+4} \partial_\xi \widehat{u_0}\|_2 < \infty$ since condition (3.4.5) is met with $\eta = b+4$ and $\nu = 1$. We also see $\||x|^{b+1} \partial_x^4 u_0\|_2 < \infty$ by condition (3.4.4) with $\alpha = 4$ and $\beta = b + 1$. Hence, $\|\mathcal{D}^b E_4\|_2 < \infty$.

For $\|\mathcal{D}^b E_{12}\|_2$ we are interested to see if $\partial_x \widehat{(x^4 u_0)}(\xi) \in H^s(\mathbb{R})$ for $s = 1/2$ so that we may apply Proposition 3. Using the Fourier transform, this is equivalent to showing that

$$\||x|^{1/2} \partial_x (x^4 u_0)\|_2 \leq \||x|^{9/2} \partial_x u_0\| + \text{ l.o.t.}$$

is finite, where l.o.t. means lower order terms. Again, condition (3.4.4) is met with $\alpha = 9/2$ and $\beta = 1$. Hence $\operatorname{sgn}(x) \partial_x \widehat{(x^4 u_0)}(\xi) \in H^b(\mathbb{R})$ for $b < s = 1/2$.

Similar computations confirm that all of the terms on the left hand side of (3.4.3) are finite. Futhermore, since $u \partial_x u \in \dot{Z}_{6,6^-}$ we may apply analagous arguments to show that the L^2-norms of the integral terms from the Duhamel formula (3.2.2) are finite. $\square$

Appendix A

Existence of infinitely many data that satisfy given conditions of Theorem 1

We will show that there exists data in the Schwartz class $u_0 \in \mathcal{S}(\mathbb{R})$ such that the following conditions hold (see equations (3.2.8) and (3.3.6)):

$$\Phi_2(u_0, t) = 4t^2 I_3(u_0) + 6t \int x u_0^2(x) dx + 12 \int x^2 u_0(x) dx = 0,$$

and

$$\Phi_1(u_0, t) = \int x u_0(x) dx + \frac{t}{4} \int u_0^2(x) dx = 0,$$

for some time, $t^* \neq 0$ where

$$I_3(u_0) = \int |D^{1/2} u_0(x)|^2 + \frac{u_0(x)^3}{3} \, dx.$$

Consider linearly independent Schwartz class functions, $f_i \in \mathcal{S}(\mathbb{R})$ for $i = 1,2,3,4$ such that the following are true:

$$\begin{array}{lll} \int f_1(x)dx = 1, & \int xf_1(x)dx = 0, & \int x^2 f_1(x)dx = 0, \\ \int f_2(x)dx = 0, & \int xf_2(x)dx = 1, & \int x^2 f_2(x)dx = 0, \\ \int f_3(x)dx = 0, & \int xf_3(x)dx = 0, & \int x^2 f_3(x)dx = 1. \end{array}$$

An application of the Paley-Weiner theorem will guarantee that such functions satisfying these conditions exist. Next, we define $\Phi_i(x) : \mathbb{R}^5 \longrightarrow \mathbb{R}$ for $i = 0,1,2$ as:

$$\Phi_0(\alpha_1, \alpha_2, \alpha_3, \alpha_4, t) = \int \sum_{j=1}^4 \alpha_j f_j \; dx$$

$$\Phi_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4, t) = \int x \left(\sum_{j=1}^4 \alpha_j f_j \right) \; dx + \frac{t}{4} \int \left(\sum_{j=1}^4 \alpha_j f_j \right)^2 dx$$

$$\begin{aligned} \Phi_2(\alpha_1, \alpha_2, \alpha_3, \alpha_4, t) = 8t^2 I_3 \left(\sum_{j=1}^4 \alpha_j f_j \right) + 6t \int x \left(\sum_{j=1}^4 \alpha_j f_j \right)^2 dx \\ +12 \int x^2 \left(\sum_{j=1}^4 \alpha_j f_j \right) dx. \end{aligned}$$

We apply the Implicit Function Theorem (IFT) to $F : \mathbb{R}^5 \longrightarrow \mathbb{R}^3$ where

$$F(\alpha_1, \alpha_2, \alpha_3, \alpha_4, t) = (\Phi_0, \Phi_1, \Phi_2).$$

We observe that $F(\vec{0}) = \vec{0}$ and that

$$DF(\vec{0}) = \left(\frac{\partial \Phi_i}{\partial \alpha_j} \right) \Bigg|_{(\alpha_1, \alpha_2, \alpha_3 \alpha_4, t) = \vec{0}} = \left(\begin{array}{ccc|cc} 1 & 0 & 0 & * & * \\ 0 & 1 & 0 & * & * \\ 0 & 0 & 12 & * & * \end{array} \right),$$

where the asterisks, $*$, denote values that are not relevant for the application of the IFT. Thus we can solve for α_j for $j = 1,2,3$ implicitly in terms of α_4 and t

in a neighborhood of the origin, i.e.,

$$F(\alpha_1(\alpha_4, t), \alpha_2(\alpha_4, t), \alpha_3(\alpha_4, t), \alpha_4, t) = \vec{0},$$

for values of α_4 and t that are sufficiently small. Furthermore, *there are infinitely many data* $u_0 = \sum_{i=1}^{4} \alpha_i f_i$ such that

$$\widehat{u_0}(0) = 0,$$

$$\Psi_1(u_0, t^*) = 6t\delta(\xi)\Phi_1(u_0, t^*) = 0,$$

and

$$\Psi_2(u_0, t^*) = it^2\delta(\xi)\Phi_2(u_0, t^*) = 0,$$

which is the desired result. □

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